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Decoherence and Triorthogonal Decomposition

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A "decoherent" measurement is a dephasing plus tracing out plus triorthogonal decomposition. Dephasing can happen any time when a small system is coupled with a large reservoir. It is in principle reversible. But in order to have a measurement we also need the tracing out and the triorthogonal decomposition. The first requirement is observation-dependent (because environment plus apparatus plus object system eventually remain in a superposition). But the second one is an irreversible and irrelative change. In the presence of three "systems" the basis degeneracy problem disappears, i.e. there can be diagonalization only relative to an observable (the measured one).

KEY WORDS: measurement; decoherence; dephasing; triorthogonal decomposition; tracing out; reversibility; irreversibility; unitary transformation.

1. INTRODUCTION

Decoherence is widely accepted today as a theory of dephasing, and is actually used in many areas, for instance in quantum information or in several models where a system is coupled with a reservoir. It does not suppose as such a departure from the reversible dynamics of elementary quantum systems. But, as a solution of the measurement problem, does not find a universal agreement. In fact, it is considered as a solution, which is point of view-dependent. Tracing out is only a relative solution and one chooses a specific form (model) of the interaction Hamiltonian between environment and object system. Here I show that decoherence has an objective character if triorthogonal decomposition is considered.

2. REQUIREMENTS OF MEASUREMENT

Busch *et al.* (1995, p. 40) have pointed out that a measurement should satisfy:

- A probability reproducibility requirement: A pointer observable $\hat{O}^{\mathcal{A}}$ must reproduce the probability measure of obtaining a result in the subset $\mathcal{X}$:

$$\wp^{\hat{O}^{\mathcal{S}}}_{\hat{\rho}^{\mathcal{S}}}(\mathcal{X}) = \wp^{\hat{O}^{\mathcal{A}}}_{\mathrm{Tr}_{\mathcal{S}}[\mathcal{T}(\hat{\rho}^{\mathcal{S}}\otimes\hat{\rho}^{\mathcal{A}})]}(f^{-1}(\mathcal{X})) \tag{1}$$

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for any value set $\mathcal{X} \in$ the spectrum $\Upsilon_{\mathcal{S}}$ of values of the measured observable and for all possible initial states $\hat{\rho}^{\mathcal{S}}$ of the system. $\mathcal{T}$ is a transform, expressing the measurement process, of the initial state $\hat{\rho}^{\mathcal{S}} \otimes \hat{\rho}^{\mathcal{A}}$ of the system $\mathcal{S}$ and the apparatus $\mathcal{A}$. The transformation $\mathcal{T}$ is called an *operation* (Davies, 1976, pp. 17–18). An operation $\mathcal{T}$ is the positive linear mapping from a state space to another, which satisfies following irreversible state transition:

$$0 \leq \mathrm{Tr}(\mathcal{T}\hat{\rho}) \leq \mathrm{Tr}(\hat{\rho}) \tag{2}$$

- An objectification requirement (of the pointer and of the values): A measurement must lead to a definite result (Busch *et al.*, 1991, p. 30). This requirement entails firstly the *pointer objectification* and secondly the *value objectification*. We shall only discuss the pointer objectification. The interpretation that $\hat{\rho}_f^{\mathcal{A}}(j, \hat{\rho}^{\mathcal{S}})$ is the final state assumed by $\mathcal{A}$ on the condition that the pointer has its value in $f^{-1}(\mathcal{X}_j)$ follows if *pointer objectification* is postulated (Busch *et al.*, 1991, pp. 36–37). The pointer objectification requires
–that the state $\hat{\rho}_f^{\mathcal{A}}(\Upsilon_{\mathcal{S}}, \hat{\rho}^{\mathcal{S}})$, for all initial states $\hat{\rho}^{\mathcal{S}}$, is a mixture

$$\mathrm{Tr}_{\mathcal{S}}\left[\mathcal{T}\left(\hat{\rho}^{\mathcal{S}} \otimes \hat{\rho}_f^{\mathcal{A}}\right)\right] = \sum_j \wp_{\hat{\rho}^{\mathcal{S}}}^{\hat{O}^{\mathcal{S}}}(\mathcal{X}_j)\hat{\rho}_f^{\mathcal{A}}(j, \hat{\rho}^{\mathcal{S}}) \tag{3}$$

of the pointer eigenstates $\hat{\rho}_f^{\mathcal{A}}(j, \hat{\rho}^{\mathcal{S}})$,
–and that

$$\hat{O}_j^{\mathcal{A}}\hat{\rho}_f^{\mathcal{A}}(j, \hat{\rho}^{\mathcal{S}}) = \hat{\rho}_f^{\mathcal{A}}(j, \hat{\rho}^{\mathcal{S}}) \tag{4}$$

This may be called *the pointer value-definiteness* condition—with respect to a reading scale $\mathbf{R}_{\mathcal{M}}$ (i.e. $\hat{O}^{\mathcal{A}} = \cup f^{-1}(\mathcal{X}_j)$).

3. NECESSARY AND SUFFICIENT CONDITIONS OF MEASUREMENT

In the following I shall show evidence for the following proposition

Proposition 3.1. (Measurement) *Necessary and sufficient condition in order to satisfy the probability reproducibility condition and the objectification requirement are the following:*

$$(\mathrm{NA} \wedge \mathrm{T1} \wedge \mathrm{T2} \wedge \mathrm{T3} \wedge \mathrm{T4}) \leftrightarrow (\mathrm{PR} \wedge \mathrm{O}), \tag{5}$$

where NA stays for 'necessity of the apparatus,' T for 'theorem,' PR for 'probability reproducibility condition' and O for 'objectification requirement.'

4. ARGUMENTATION

4.1. First Requirement

Let us discuss these requirements in details. The first requirement is a general one: the necessity of the apparatus. It is also valid for classical mechanics. In quantum mechanics there is further problem that if we wish to measure observables which commute with the energy, and the system is an initial state of superposition of eigenstates of the energy, then there is no means to obtain by unitary evolution an eigenstate of the measured observable and then a determined property.

4.2. Second Requirement

The second requirement is the following. We need that the apparatus $\mathcal{A}$ and the system $\mathcal{S}$ can be coupled in such a way that there is a relationship between properties of the system and values of the pointer. In other words the total wave function of the system $\mathcal{S}+$ the apparatus $\mathcal{A}$ should be represented after the interaction as a superposition of pairs of subsystem states such that there is a shift of the pointer observable from the initial value, which would allow the storage of the result of a measurement in the memory of $\mathcal{A}$. If we let $\mathcal{S}$ and $\mathcal{A}$ interact, so as to measure some observable of $\mathcal{S}$ from a time $t = 0$ to t_s (where interaction stops), then we could write the Schrödinger equation of the whole system as follows:

$$\imath\hbar\frac{\partial}{\partial t}\left|\Psi_t^{\mathcal{S}+A}\right\rangle = \hat{H}_{\mathcal{S}A}\left|\Psi_t^{\mathcal{S}+A}\right\rangle. \tag{6}$$

If we explicitly write the dependence of the wave functions of the two subsystems on the position: $\varsigma(\mathbf{q})$, $\mathcal{A}(\mathbf{r})$—since measurement with an apparatus can be seen as a position of a pointer on a graduate scale, then the state:

$$\Psi_t^{\mathcal{S}+A}(\mathbf{q},\mathbf{r}) = \varsigma(\mathbf{q})\mathcal{A}(\mathbf{r}-\mathbf{q}t) \tag{7}$$

is a solution of the equation

$$\imath\hbar\frac{\partial}{\partial t}\Psi_t^{\mathcal{S}+A}(\mathbf{q},\mathbf{r}) = \hat{H}_{\mathcal{S}A}\Psi_t^{\mathcal{S}+A}(\mathbf{q},\mathbf{r}). \tag{8}$$

Obviously, this is not a general formulation. It depends on the form of the interaction Hamiltonian. But any measurement presents specific constraints, which are dependent on the observable one chooses to measure and on the system one chooses to measure on.

Consider now the time $t = t_s$, at which interaction stops, when there is no more definite independent apparatus state. Then we can formulate the following theorem.

Theorem 4.1. (Everett) *The total wave function of the system + apparatus can be represented after the interaction as a superposition of pairs of subsystem states* *of the form:*

$$\Psi_{t_s}^{\mathcal{S}+A}(\mathbf{q}, \mathbf{r}) = \int \varsigma(\mathbf{q}')\delta^3(\mathbf{q}-\mathbf{q}')\mathcal{A}(\mathbf{r}-\mathbf{q}t_s)d\mathbf{q}' \tag{9}$$

where the term $\mathcal{A}(\mathbf{r}-\mathbf{q}t_s)$ expresses a shift of $\mathbf{q}t_s$ from the initial $\mathcal{A}(\mathbf{r})$ which allows the storage of the result of a measurement in the memory of $\mathcal{A}$.

However, as we know (by the objectification requirement), the state resulting from a measurement process can neither be an entanglement of $\mathcal{A}$ and $\mathcal{S}$, nor a superposition of $\mathcal{S}$'s or $\mathcal{A}$'s states, but needs to be a (classical) statistical mixture (for instance, a statistical mixture describes the probabilities of outcomes of a classical dice before the result is read). But not all forms of mixture of states of a system $\mathcal{S}$ and an apparatus $\mathcal{A}$ are good for describing the final state resulting from a measurement process. As we know by Everett theorem, the necessary condition of a measurement is that the density matrix of the apparatus changes during the measurement process, i.e. $\mathcal{A}$ extracts information from $\mathcal{S}$ (Joos, 1996a, p. 44). We can synthesize these two requirements in short as:

$$\hat{\rho}^{\mathcal{S}} \rightsquigarrow \sum_j \hat{P}_j \hat{\rho}^{\mathcal{S}} \hat{P}_j \tag{10}$$

for the different eigenvalues j of the measured observable $\hat{O}^{\mathcal{S}}$. In fact, suppose, for the sake of simplicity, that $\hat{\rho}^{\mathcal{S}} = |\psi\rangle\langle\psi|$ and $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$, where $|0\rangle$ and $|1\rangle$ are the eigenkets of the measured observable $\hat{O}^{\mathcal{S}}$. Then, it is straightforward that

$$\begin{aligned}
\sum_{j=0}^{1} \hat{P}_j \hat{\rho}^{\mathcal{S}} \hat{P}_j &= |0\rangle\langle 0| \left(c_0|0\rangle + c_1|1\rangle\right) \left(\langle 0|c_0^* + \langle 1|c_1^*\right) |0\rangle\langle 0| \\
&\quad + |1\rangle\langle 1| \left(c_0|0\rangle + c_1|1\rangle\right) \left(\langle 0|c_0^* + \langle 1|c_1^*\right) |1\rangle\langle 1| \\
&= \left(c_0|0\rangle\langle 0|0\rangle + c_1|0\rangle\langle 0|1\rangle\right) \left(c_0^*\langle 0|0\rangle\langle 0| + c_1^*\langle 1|0\rangle\langle 0|\right) \\
&\quad + \left(c_0|1\rangle\langle 1|0\rangle + c_1|1\rangle\langle 1|1\rangle\right) \left(c_0^*\langle 0|1\rangle\langle 1| + c_1^*\langle 1|1\rangle\langle 1|\right) \\
&= |c_0|^2|0\rangle\langle 0| + |c_1|^2|1\rangle\langle 1| = |c_0|^2 \hat{P}_0 + |c_1|^2 \hat{P}_1,
\end{aligned} \tag{11}$$

which is the desired result. Such a mixture is called a *Lüders mixture* (Lüders, 1951).

4.3. Third Requirement

The third requirement is that either the value of the macroscopic pointer reduces to an eigenvalue of a quantum observable or it is a type of average over microscopic observables.

In fact, the assumption that the pointer observable is classical implies (Busch *et al.*, 1991, pp. 82–83; Mittelstaedt, 1998, p. 109).

Theorem 4.2. (Classicality of pointer) *Let a measurement scheme* $\langle \mathcal{H}_A, \hat{O}^{\mathcal{A}}, |\mathcal{A}\rangle, \hat{U}\rangle$ *be a candidate for a measurement of a discrete sharp observable* $\hat{O}^{\mathcal{S}}$*, where* $\mathcal{H}_A$ *is the Hilbert space of the apparatus,* $|\mathcal{A}\rangle$ *is the ket describing the apparatus and* $\hat{U} = e^{\imath \hat{H}_{\mathcal{S}A}\tau}$*, where* $\hat{H}_{\mathcal{S}A}$ *is the interaction Hamiltonian between system and apparatus and* τ *is the time interval of interaction. If* $\hat{O}^{\mathcal{A}}$ *is classical, then the coupling g cannot be generated by an observable of* $\mathcal{S} + A$*, because in this case* $\langle \mathcal{H}_A, \hat{O}^{\mathcal{A}}, |\mathcal{A}\rangle, \hat{U}\rangle$ *cannot fulfill the probability reproducibility condition (1).*

Proof: Suppose that pointer observable $\hat{O}^{\mathcal{A}}$, referred to a discrete sharp observable $\hat{O}^{\mathcal{S}}$ of $\mathcal{S}$, is classical. We assume that $\hat{H}_{\mathcal{S}A}$, the interaction Hamiltonian between system and apparatus, commutes with $\hat{O}^{\mathcal{A}}$. In fact, an interaction Hamiltonian commutes with all classical observables pertaining to the coupled systems. Therefore, $\hat{O}^{\mathcal{A}}$ also commutes with $\hat{U}$ and it follows that the probability distribution of $\hat{O}^{\mathcal{A}}$ is completely independent from the measured observable. In other words, the apparatus remains uncoupled with the object system. We can show it by taking the mean value of $\hat{O}^{\mathcal{A}}$:

$$\begin{aligned}\langle \Psi(\tau)|\hat{O}^{\mathcal{A}}|\Psi(\tau)\rangle &= \langle \hat{U}\Psi|\hat{O}^{\mathcal{A}}|\hat{U}\Psi\rangle = \langle \Psi|\hat{U}^{\dagger}\hat{O}^{\mathcal{A}}\hat{U}|\Psi\rangle \\ &= \langle \Psi|\hat{U}^{\dagger}\hat{U}\hat{O}^{\mathcal{A}}|\Psi\rangle \\ &= \langle \Psi|\hat{O}^{\mathcal{A}}|\Psi\rangle, \end{aligned} \tag{12}$$

where $|\Psi\rangle = |\varsigma\rangle|\mathcal{A}\rangle$, $|\varsigma\rangle$ is the ket describing the object system and $\langle \Psi(\tau)| = \hat{U}\langle \Psi|$. Such a situation is incompatible with condition (1) unless $\hat{O}^{\mathcal{S}}$ is trivial (i.e. constant). QED. □

Therefore the pointer cannot really be classical. But, since it also seems that it cannot be quantum mechanical (due to Everett theorem), then Machida and Namiki proposed what follows.

Proposition 4.2. (Machida/Namiki) *The value of a macroscopic variable of* $\mathcal{A}$ *read in a measurement is not an eigenvalue of a QM observable but is a kind of average over microscopic observables, i.e. over a large number of Hilbert spaces.*

In other words, for Machida and Namiki, $\mathcal{A}$—we mean here not only the little measuring device, but also the amplifiers—cannot be considered as a proper quantum system if we consider the final macroscopical result. Machida and Namiki have presented a model of how it can happen. More details on the model can be found in (Machida and Namiki, 1980, pp. 1837–1839).[2] The Machida/Namiki model is very interesting because it acknowledges the decoherence as in general imperfect, i.e. such that we never have elimination of off-diagonal terms (Namiki and Pascazio, p. 321). But, on the other hand, there is a risk of interpreting the measurement process exclusively in statistical terms (not only the measuring apparatus, especially the amplifying device, but also the object system $\mathcal{S}$ itself), by excluding a description of individual events.[3]

Furthermore, the statistical character of the model presents some difficulties also for the pointer. In fact, the essential aspect of a measurement is the change in the pointer's position in the apparatus, whereas the scattering process of the Machida/Namiki model would leave the scatterer practically unchanged (if measured by the inner product)—dephasing is not a transformation. Hence the model can only account for ensembles and not for individual measurements. However, a recent correction due to Nakazato/Pascazio (Nakazato and Pascazio, 1993) takes into account the energy exchange between the system and the apparatus.

4.4. Fourth Requirement

The fourth requirement is that there is an environment $\mathcal{E}$ which makes all information about the premeasured system unavailable with only one exception. Previously the influence of $\mathcal{E}$ was normally considered a noise factor. Instead, we can understand the importance of this factor by formulating the following theorem (which synthesizes the analysis of both Zeh and Zurek).

Theorem 4.3. (Zeh/Zurek) *The Environment washes out all information about the premeasured system with only one exception: when the Hamiltonian $\hat{H}_{\mathcal{AS}}$, which couples $\mathcal{A}$ and $\mathcal{S}$, commutes with an observable $\hat{O}^{\mathcal{S}}$:*

$$[\hat{O}^{\mathcal{S}}, \hat{H}_{\mathcal{AS}}] = 0 \tag{13}$$

then this particular observable will not be disturbed, so that the pointer of the apparatus will contain the information about this observable and only this one.

The theorem guarantees the required correlation between the system's and the apparatus' observables. Obviously, such a theorem cannot be understood in the sense of a classicality of the pointer, due to the problem seen above (Theorem 2). In fact, as we shall discuss below, interference terms are never destroyed and other observables which do not commute with the one measured, always enter, to a certain extent, in the result which is never 100% 'determined'—in other words, we have here a POVM. We also have the following consequence of the previous theorem for the pointer observable:

[2] See also (Auletta, 2000, pp. 279–281).
[3] As is acknowledged by Namiki and Pascazio (1993, p. 325).

Corollary 4.1. (Zurek) *The pointer observable $\hat{O}^{\mathcal{A}}$ commutes with the Hamiltonian $\hat{H}_{\mathcal{A}S}$:*

$$[\hat{O}^{\mathcal{A}}, \hat{H}_{\mathcal{A}S}] = 0 \tag{14}$$

4.5. Fifth Requirement

Finally, the fifth requirement is that the transformation (state reduction) is not unitary but is obtained by tracing out the environment. In fact, we have said that we need a Lüders mixture. Now, if the initial state of $\mathcal{A} + S$ is a pure state, there is no means to obtain a mixture by unitary evolution. In fact a pure state remains a pure state after a unitary evolution, as can be easily proved.

Proof: We start with the equality for pure states:

$$\begin{aligned} \hat{\rho}_t^2 &= \hat{U}_t \hat{\rho}_0 \hat{U}_t^\dagger \hat{U}_t \hat{\rho}_0 \hat{U}_t^\dagger \\ &= \hat{U}_t \hat{\rho}_0 \hat{U}_t^\dagger = \hat{\rho}_t \end{aligned} \tag{15}$$

so that if $\hat{\rho}_0^2 = \hat{\rho}_0$, then we also have: $\hat{\rho}_t^2 = \hat{\rho}_t$, i.e., after a unitary and continuous evolution, we again have a pure state and not a mixture. QED. □

Hence we can summarize the results of Shea/Scully/McCullen (Scully, *et al.*, 1978; Zurek, 1981, 1982; Cuini, 1983), in the following theorem.

Theorem 4.4. (Partial trace) *The reduction required by a measurement can be obtained with a partial trace, which is not unitary.*

The result of such a process is a mixture generated by a partial trace with respect to a total system, which eventually remains in a pure state.

Therefore, summarizing the analysis of Zeh (1970, 1993) and Joos (1996a, pp. 43–44, 115–124), we can formulate following theorem.

Theorem 4.5. (Joos/Zeh) *Off-diagonal terms of the density matrix describing the object system, cannot be destroyed in the $\mathcal{S} + A$ system as a whole, but only downloaded into the environment.*

This theorem guarantees the necessary correlation between values. Finally the downloading of off-diagonal terms in the environment implies following corollary as a consequence:

Corollary 4.2. (Off-diagonal terms) *The off-diagonal terms of the $\hat{\rho}$ of the object system tend to zero in a measurement process but they can never really be zero.*

5. DISCUSSION

One could say that, being a measurement the result of a partial tracing out, it is only relative to the observer or to a particular point of view (this is the point of the many-world interpretation). But it is not so. In fact, the fourth requirement introduces three systems in interaction. Now, when apparatus and object system interact alone, there is always a basis degeneracy. In fact, let us write (Zurek, 1981, p. 1516, 1982)

$$|\mathcal{A}_0\rangle \otimes |\varsigma\rangle \mapsto \sum_o c_o |\mathcal{A}_o\rangle \otimes |o\rangle, \tag{16}$$

where one could say that we have measured observable $\hat{O}$ for which the spectral decomposition is $\hat{O} = \sum_o c'_o |o\rangle\langle o|$. But suppose that we express the state of $\mathcal{A}$ in another basis $\{|\mathcal{A}_{o'}\rangle\}$ composed of superposition of states $|\mathcal{A}_o\rangle$:

$$|\mathcal{A}_{o'}\rangle = \sum_o \langle \mathcal{A}_o | \mathcal{A}_{o'}\rangle |\mathcal{A}_o\rangle. \tag{17}$$

Then we can rewrite state (16) of the compound system as:

$$\sum_o c_o |\mathcal{A}_o\rangle \otimes |o\rangle = \sum_{o'} |\mathcal{A}_{o'}\rangle \otimes \sum_o c_o \langle \mathcal{A}_{o'} | \mathcal{A}_o\rangle |o\rangle = \sum_{o'} c''_{o'} |\mathcal{A}_{o'}\rangle \otimes |o'\rangle. \tag{18}$$

If the coefficients c_o in Equation (16) have the same magnitude, then (by RHS of Equation (18)) whenever the set $\{|\mathcal{A}_{o'}\rangle\}$ is orthonormal, also the set $\{|o'\rangle\}$ is orthonormal. Then $\mathcal{A}$ contains not only information about the observable $\hat{O}$ but also about another observable $\hat{O}' = c'''_{o'} |o'\rangle\langle o'|$, even if normally $\hat{O}$ and $\hat{O}'$ do not commute. And the same can be said about many other observables. We call such a problem the *basis degeneracy problem*.

But this is not so in the case of a triorthogonal decomposition (Elby and Bub, 1994, pp. 4215–4216). First, let us state the following lemma.

Lemma 5.1. ***(Elby/Bub)*** *Let $\{|a_j\rangle\}$ and $\{|s_j\rangle\}$ be linearly independent sets of vectors, respectively, in $\mathcal{H}_1, \mathcal{H}_2$ for two generic systems $\mathcal{S}_1, \mathcal{S}_2$. Let $\{|s'_j\rangle\}$ be a linearly independent set of vectors that differs non-trivially from $\{|s_j\rangle\}$. If $|\Psi\rangle = \Sigma_j c_j |a_j\rangle \otimes |s_j\rangle$, then $|\Psi\rangle = \Sigma_j c'_j |a'_j\rangle \otimes |s'_j\rangle$ only if at least one of the $\{|a'_j\rangle\}$ vectors is a linear combination of (at least two) $\{|a_j\rangle\}$ vectors.*

We omit the proof of the lemma,[4] but use it to prove *per contradictionem* the uniqueness of triorthogonal decomposition (in order to indirectly prove Theorem 3 and its corollary, so that the required pointer objectification can be obtained).

Proof: Suppose a vector $|\Psi\rangle = \Sigma_j c_j |a_j\rangle \otimes |s_j\rangle \otimes |e_j\rangle$, where $\{|a_j\rangle\}, \{|s_j\rangle\}$, $\{|e_j\rangle\}$ are orthogonal sets of vectors respectively in $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ for three generic

[4] See the original article of Elby and Bub.

systems $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$. Then, we claim that even if some of the $|c_j|$s are equal, no alternative orthogonal sets $\{|a'_j\rangle\}$, $\{|s'_j\rangle\}$, $\{|e'_j\rangle\}$ exist such that $|\Psi\rangle = |a'_j\rangle \otimes |s'_j\rangle \otimes |e'_j\rangle$, unless each alternative set of vectors differs only trivially from the set it replaces. Assume, without loss of generality, that $\{|e_j\rangle\}$ differs non-trivially from $\{|e'_j\rangle\}$, and let us write $|\Psi\rangle = \Sigma_j c_j |f_j\rangle \otimes |e_j\rangle$ (where $|f_j\rangle := |a_j\rangle \otimes |s_j\rangle$). Now supposing $|\Psi\rangle = \Sigma_j c'_j |f'_j\rangle \otimes |e'_j\rangle$ (where $|f'_j\rangle := |a'_j\rangle \otimes |s'_j\rangle$), we cannot rewrite the factorisable state $|a'_j\rangle \otimes |s'_j\rangle$ as an entangled state. □

But according to Lemma 1, since $|\Psi\rangle = \Sigma_j c_j |f_j\rangle \otimes |e_j\rangle$ and since $\{|e_j\rangle\}$ differs non-trivially from $\{|e'_j\rangle\}$, then we have $|\Psi\rangle = \Sigma_j c'_j |f'_j\rangle \otimes |e'_j\rangle$ only if $|f'_k\rangle = \Sigma_j g_{jk} |f_j\rangle$, where at least two of the g_{jk}'s are non-zero. But since $|f_j\rangle := |a_j\rangle \otimes |s_j\rangle$, it follows that $|f'_k\rangle$ is an entangled state ($|f'_k\rangle = \Sigma_j g_{jk} |a_j\rangle \otimes |s_j\rangle$), which is the required contradiction. QED.

The above proof shows that one should not identify the triorthogonal decomposition presupposed by decoherence with the von Neumann chain: in the second case, only successive measurements, in which each decomposition is biorthogonal, are considered. Note that, while the partial trace is a mathematical formalism which reflects our ignorance (our 'discarding') of the environment and that is therefore relative to the subsystem over which we perform such a partial trace, the triorthogonal decomposition—due to the introduction of a third factor (the environment)—is not a relative property (not relative to an observer) but an absolute one.

In summary, a "decoherent" measurement is therefore dephasing plus tracing out plus triorthogonal decomposition. Dephasing can happen any time when a small system is coupled with a large reservoir. It is in principle reversible. But in order to have a measurement we need also the tracing out and the triorthogonal decomposition. The first requirement is observation-dependent (because $\mathcal{S} + \mathcal{A} + \mathcal{E}$ eventually remains in a superposition); but the second one is an irreversible and irrelative change. In the presence of three "systems" ($\mathcal{S} + \mathcal{A} + \mathcal{E}$) the basis degeneracy problem disappears, i.e. there can be diagonalization only relatively to an observable (the measured one).

In other words decoherence as a theory of measurement is characterized by unsharpness plus objectivity.

6. MWI AND DECOHERENT HISTORIES

Here we wish to stress some critical points about the many-world interpretation and the decoherent histories.

- The many-world interpretation (MWI) is not consistent with the triorthogonal decomposition requirement. In fact either it is subjected to the basis degeneracy problem or must admit some form of objective 'reduction' of the wave, contrarily to its original aim.

- On the other hand, if we wish to maintain decoherence (as some proponents of the decoherent histories do), we must introduce an operation of tracing out (see Theorem 4). Now it is possible to argue that a reversibility is excluded only in the system + apparatus + environment reference 'frame,' and that, by taking larger systems or other ones, one could obtain other decompositions. Apart from the problem of what 'larger' or 'other' systems could be (see next point), if so, i.e. if we accept the second alternative (symmetry of time), then we are faced with another alternative: either we follow Griffiths, and suppose that there is a real loss of diagonal terms, or we suppose that we never have a real loss of coherence. And here, while Griffiths' proposal assumes that alternative histories pertain to a statistical mixture (without interference terms between them), the proponents of the decohering history approach seem to suppose that the alternative histories are in superposition. But if so, 'decohered' or macroscopical objects become only illusory while the decohered histories should be more imaginary than real.

  In this case then the decohering histories interpretation would be nothing more than the MWI itself and, as the MWI, would be subjected to the fallacy of the basis degeneracy.
- We have seen that trio-orthogonal decomposition is something more than the partial trace. However, we do not consider this problem here and consider only the operation of partial tracing. It seems then, that in this way one can distinguish the decoherent history approach from the MWI. And in this way Joos and Zeh, Omnès in later works (Omnès, 1994) and Kiefer (1996b, p. 177) seem to understand the subject. But how can a history be decohering if it is a history of the whole universe, and hence without an 'external environment' able to produce such a *local* and *partial* effect? Naturally it should happen relatively to us, i.e. to some particular and partial tracing out. But here the situation is inverted: we do not have one or two small systems ($\mathcal{S} + \mathcal{A}$), which interact with a large environment, but a very big environment (the whole universe!) which is 'measured' in some way by small systems *within it*.
- But, apart from the problems posed by the above inversion, is it better to suppose that it is possible to 'measure' the universe as such or some of its observables? How could it be possible to 'measure' in one way or another the wave function of the universe? By which technical and theoretical means? In fact there is no way to know directly the wave function or the state of the universe, since our universe is unique. In general, if we wish to maintain some statistical value for a wave function, then we cannot use the wave function for very big systems—and certainly not for the whole universe—because it is impossible—even in principle—to reproduce exactly the same conditions (to make identical copies of a state of our universe) (Woo, 1986, p. 924; Fink and Leschke, 2000).

- And, as a consequence, since partial tracing depends from the point of view of an observer or of an apparatus 'internal' with respect to the observed system (the universe), this signifies that there can be reduced states (by tracing out the rest of the world) of different states of the universe which cannot be distinguished by this 'internal' observer or apparatus.[5]

  For example (Breuer, 1997, p. 109) there are several 'states of the universe', which are different from one other only through EPR-correlations between several observers of the same event: these observers then 'see' the same event though the states of the whole are different.

  Generalizing, since a measurement is a bilinear correspondence between the states of a system $\mathcal{S}$ (here the environment) and the states of an apparatus $\mathcal{A}$ from which the properties of $\mathcal{S}$ can be inferred (see Theorem 1, p. 4), it follows that $\mathcal{S}$ determines $\mathcal{A}$. But, as we know, in QM a whole (the universe) in entanglement does not necessarily determines the parts (the 'measuring' apparatus). And, if we are not able to measure some observables, how can we speak in any form of the wave function of the universe?

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[5] As can be deduced from (Mittelstaedt, 1998, pp. 120–121). Breuer has shown (Breuer, 1997, pp. 53–75) that in general an 'internal' apparatus cannot distinguish all states of a system, and this result is valid in both classical and quantum mechanics.

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